Existence of Unique Solution and Hyers-Ulam stability for Fractional differential Equation with Sturm-Liouville Boundary Conditions

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Abstract: In this paper we consider the following fractional order boundary value problem with Sturm-Liouville type boundary conditions

$$^{C}D_{0^+}^{\mathrm{r}}\mathrm{u}(\mathrm{t})=\mathrm{f}(\mathrm{t},\mathrm{u}(\mathrm{t}))$$

$$au(0) = bu'(0),$$
$$cu(1) = -du'(1),$$

where $^{C}D_{0^+}^{\mathrm{r}}$ denotes Caputo fractional derivatives of order $1 < \mathrm{r} \leq 2$, and established existence of unique solution by Banach contraction mapping therem and also studied Hyers-Ulam stability.

Keywords: fractional order boundary value problem, Banach contraction, fixed point theorem

1 Introduction

Fractional differential equations has recently attracted many researchers due to its wide applications [9, 11, 26] in engineering, technology, biology and so on. Establishing existence of solutions for fractional differential equations together with boundary conditions has been carried out by researchers [8, 12, 13, 14, 15, 23, 24, 25].

In the recent years, the study of differential equations of fractional order has been in the limelight by many researchers in the areas of applied sciences, such as engineering, physics, biology and economics. This is basically because, it finds applications in several real world problems. For details on the theory and some applications of fractional differential equations, see the monographs of [11, 16, 19, 26]. In the qualitative theory of differential equations, various theorems have been extensively deployed by researchers in establishing the existence and uniqueness of solutions to both the initial and boundary value problems. In [22], Srivastava et al. studied the hybrid fixed point theorems of Krasnosel'skii type, which involve product of two operators in partially ordered normed linear spaces and applied to fractional integral equations for establishing the existence of solutions under certain monotonicity conditions blending with the existence of the upper or lower solution.

Babakhani and Baleanu [5] considered the following nonlinear fractional order differential equations

$$\begin{aligned}(D^{\alpha} - \rho t D^{\beta})(\mathrm{z}(t)) &= f(t, \mathrm{z}(t), D^{\gamma}(\mathrm{z}(t))),\ t \in (0,1),\ 1 < \alpha \leq 2,\ 0 < \beta + \gamma \leq \alpha, \\ \mathrm{z}(0) &= \mathrm{z}_0, \mathrm{z}(1) = \mathrm{z}_1,\end{aligned} \quad (1.1)$$

also studied

$$\begin{aligned}&(D^{\alpha} - \rho t D^{\beta})(\mathrm{z}(t)) = f(t, \mathrm{z}(t), D^{\gamma}(\mathrm{z}(t))),\ t \in (0,1),\ 1 < \alpha \leq 2,\ 0 < \beta + \gamma \leq \alpha, \\ &\mathrm{z}(0) = \mathrm{z}_0, \mathrm{z}'(0) = 1,\end{aligned} \quad (1.2)$$

and derived necessary conditions for the existence of solutions for (1.1) and (1.2). Recently, Anber et al. [4] studied the infinite system of fractional order two-point boundary value problem

$$\begin{aligned}&D^{\alpha}(\mathrm{z}(t)) + f(t, \mathrm{z}(t)),\ t \in (0,1),\ \alpha \in (1,2), \\ &\mathrm{z}(0) = \int_0^1 g(s)\mathrm{z}(s)ds, \mathrm{z}(1) = 0,\end{aligned} \quad (1.3)$$

and established existence of solutions.

Fixed point theory is one of the most important area of research in Mathematics. In recent years, many results related to fixed point theorems in ordered metric spaces are established in [1, 2, 3, 18, 21] and etc. The results in this line was obtained by Ran and Reurings[20]. Subsequently, Nieto and Rodriguez-Lopez[17] extended the results by omitting the continuity hypothesis and applied their result to obtain a unique solution to a first order ordinary differential equation with periodic boundary conditions. Latter, Bhaskar and Lakshmikantham [7] established several coupled fixed point theorems for mixed monotone mappings defined on partially ordered complete metric spaces.

Motivated by the works mentioned above, in this paper we study the following fractional order differential equation

$$\begin{aligned} &{}^{C}D_{0^+}^{\mathrm{r}}\mathrm{u(t)}=\mathrm{f(t,u(t))} \\ &au(0) = bu'(0), \\ &cu(1) = -du'(1), \end{aligned} \tag{1.4}$$

where ${}^{C}D_{0^+}^{r}$ is left sided Caputo fractional derivative of order $r, 1 < r \leq 2, f \in C([0,1] \times [0,+\infty), [0,+\infty))$, a, b, c, d are real positive constants and established necessary conditions for the existence of solutions by applying Banach contraction mapping theorem.

2 Preliminaries

In this section, we provide some definitions and lemmas which are needed in the latter discussion.

Definition 2.1 [9] Let $\alpha \in (0,+\infty)$. The operator $I_{a^+}^{\gamma}$ defined on $L_1[a,b]$ by

$$I_{a^+}^{\gamma} f(t) := \frac{1}{\Gamma(\gamma)} \int_a^t (t-s)^{\gamma-1} f(s) ds,$$

for $t \in [a,b]$, is called the left sided Riemann-Liouville fractional integral of order γ. Under same hypotheses, the right-sided Riemann-Liouville fractional integral operator is given by

$${}_{b^-}I^{\gamma} f(t) := \frac{1}{\Gamma(\gamma)} \int_t^b (s-t)^{\gamma-1} f(s) ds.$$

Definition 2.2. [9] Suppose $\gamma > 0$ with $n = [\gamma] + 1$. Then the left and right sided Caputo fractional derivatives defined on absolutely continuous functions space $AC^n[a,b]$ are given by

$$({}^{C}D_{a^+}^{\gamma} f)(t) = (I_{a^+}^{n-\gamma} D^n f)(t), \qquad ({}_{b^-}^{C}D^{\gamma} f)(t) = (-1)^n ({}_{b^-}I^{n-\gamma} D^n f)(t),$$

where $D^n := \frac{d^n}{dt^n}$.

Theorem 2.3. [Banach, [6]] Let X be a nonempty set and let d be a metric on X such that (X,d) forms a complete metric space. If the mapping $F: X \to X$ satisfies

$$d(Fu, Fv) \leq kd(u,v) \quad \text{for some} \quad 0 < k < 1 \quad \text{for all} \quad u, v \in X,$$

then there is a unique $w \in X$ such that $Fw = w$.

Definition 2.4 [10] The problem (1.4) is called Hyers-Ulam stable whenever there exists a real constant > 0 such that, for each $\varepsilon > 0$ and $\mathrm{z}(t) \in C([0,1], \mathbb{R})$ satifying

$$\left| {}^{C}D_{0^+}^{r} u(t) - f(t, u(t)) \right| < \varepsilon \quad \text{for all} \quad t \in [0,1],$$

there is a solution $\hat{\mathrm{u}}(t) \in C([0,1], \mathbb{R})$ of problem (1.4) such that

$$|\mathrm{z}(t) - \hat{\mathrm{u}}(t)| < \varepsilon \quad \text{for all} \quad t \in [0,1].$$

Theorem 2.5. Let $V \in C([0,1], \mathbb{R})$. Then the boundary value problem

$${}^{C}D_{0^+}^{r}(u(t)) + V(t) = 0, 0 < t < 1, \tag{2.1}$$

$$\begin{aligned} au(0) &= bu'(0), \\ cu(1) &= -du'(1), \end{aligned} \tag{2.2}$$

has a unique solution

$$u(t) = \int_0^1 \mathcal{K}(t,s) V(s) ds, \tag{2.3}$$

where

$$\mathcal{K}(t,s) := \begin{cases} \mathcal{K}_1(t,s), & 0 \leq s \leq t \leq 1,1.2mm \\ \mathcal{K}_2(t,s), & 0 \leq t \leq s \leq 1, \end{cases} \tag{2.4}$$

and

$$\mathcal{K}_1(t,s) = G_2(t,s) - \frac{(t-s)^{r-1}}{\Gamma(r)},$$

$$\mathcal{K}_2(t,s) = \frac{\Delta}{\Gamma(r)} [c(1-s)^{r-1} + d(r-1)(1-s)^{r-2}](at+b),$$

and $\Delta = (ac + ad + bc)^{-1}$.

Proof. The equivalent integral equation to (2.1) is given by

$$u(t) = C_1 + C_2 t - \int_0^t \frac{(t-s)^{r-1}}{\Gamma(r)} V(s)ds.$$

By the boundary conditions (2.2), we can determined A and B as

$$C_1 = \frac{\Delta b}{\Gamma(r)} \int_0^1 [c(1-s)^{r-1} + d(r-1)(1-s)^{r-2}]V(s)ds,$$

$$C_2 = \frac{\Delta a}{\Gamma(r)} \int_0^1 [c(1-s)^{r-1} + d(r-1)(1-s)^{r-2}]V(s)ds.$$

Thus, we have

$$\begin{aligned} u(t) \quad &= \frac{\Delta}{\Gamma(r)} \int_0^1 [c(1-s)^{r-1} + d(r-1)(1-s)^{r-2}](at+b)V(s)ds \\ &5cm - \frac{1}{\Gamma(r)} \int_0^t (t-s)^{r-1} V(s)ds \\ &= \int_0^1 \mathcal{K}(t,s)V(s)ds. \end{aligned}$$

Therefore,

$$u(t) = \int_0^1 \mathcal{K}(t,s)V(s)ds.$$

Theoremn 2.6. The Green's function $\mathcal{K}(t,s)$ has the following properties:

(i) $\mathcal{K}(t,s)$ is continuous on $[0,1] \times [0,1]$,

(ii) for $r > \frac{2a+b}{a+b}$, we have $\mathcal{K}(t,s) > 0$ for any $t,s \in [0,1]$,

(iii) for $r > \frac{2a+b}{a+b}$, we have $\mathcal{K}(t,s) \le \mathcal{K}(s,s)$ for $t,s \in [0,1]$,

(iv) $\mathcal{K}(s,s) \le \mathcal{K}(t,s)$ for $t,s \in [0,1]$, where

$$= \min\left\{\frac{bd(r-1)-a(c+d)}{bc+bd(r-1)}, \frac{b}{a+b}\right\}.$$

Proof. (i) is evident. We prove (ii).

For $0 \le s \le t \le 1$, we have

$$\frac{\partial \mathcal{K}_1(t,s)}{\partial t} = \frac{a\Delta}{\Gamma(r)}[c(1-s)^{r-1} + d(r-1)(1-s)^{r-2}] - \frac{(r-1)(t-s)^{r-2}}{\Gamma(r)}$$

and

$$\frac{\partial^2 \mathcal{K}_1(t,s)}{\partial t^2} = \frac{(r-1)(2-r)(t-s)^{r-3}}{\Gamma(r)} \ge 0.$$

This shows that $\frac{\partial \mathcal{K}_1(t,s)}{\partial t}$ is increasing on $t \in [s,1]$. So by $r > \frac{2a+b}{a+b}$, we have

$$\begin{aligned} \frac{\partial \mathcal{K}_1(t,s)}{\partial t} \quad &\le \frac{\partial \mathcal{K}_1(1,s)}{\partial t} \\ &= \frac{a\Delta}{\Gamma(r)}[c(1-s)^{r-1} + d(r-1)(1-s)^{r-2}] - \frac{(r-1)(1-s)^{r-2}}{\Gamma(r)} \\ &\le \frac{ac\Delta + (ad\Delta - 1)(r-1)(1-s)^{r-2}}{\Gamma(r)} \le 0. \end{aligned}$$

Then $\mathcal{K}_1(t,s)$ is decreasing with respect to t on $[s,1]$, we get

$$\mathcal{K}_1(1,s) \le G_1(t,s) \le \mathcal{K}_1(s,s).$$

Similarly, we can have

$$0 < \mathcal{K}_2(0,s) \le \mathcal{K}_2(t,s) \le \mathcal{K}_2(s,s).$$

From the proof of (ii), we have $\mathcal{K}(t,s) \le G(s,s)$. Moreover, we have

$$\kappa(s) \le \mathcal{K}(t,s) \le G(s,s),$$

where

$$\kappa(s) := \begin{cases} \mathcal{K}_1(1,s), 0.6cm 0 \le s < \frac{ad(2-r)+bc}{ad+bc}, 1.2mm \\ G_2(0,s), 0.6cm \frac{ad(2-r)+bc}{ad+bc} \le s < 1. \end{cases}$$

Since

$$\frac{\mathcal{K}_1(1,s)}{\mathcal{K}(s,s)} \le \frac{bd(r-1)-a(c+d)}{bc+bd(r-1)}$$

and

$$\frac{\mathcal{K}_2(0,s)}{G(s,s)} = \frac{b}{as+b} \geq \frac{b}{a+b},$$

This completes the proof.

3 Main Results

In this section we establish the existence and uniqueness of solutions by an application of fixed point approaches.

Theorem 3.1. Let L be a nonnegative constant such that

$$|f(t,\mathrm{u}) - f(t,\mathrm{v})| \leq M|\mathrm{u}-\mathrm{v}| \quad \forall \quad (t,\mathrm{u}),(t,\mathrm{v}) \in [0,1]\times\mathbb{R}.$$

$$M\int_0^1 |\mathcal{K}(s,\mathrm{s})|ds < 1, \tag{3.1}$$

then the BVP (1.4) has a uniqe nontrivial solution in $C[0,1]$.

Proof. Consider the operator $\mathcal{P}: C[0,1] \to C[0,1]$ defined by

$$(\mathcal{P}u)(t) = \int_0^1 \mathcal{K}(t,\mathrm{s})f(\mathrm{s},u(\mathrm{s}))ds.$$

In view of (2.3) we wish to show that there exists a unique $u \in C([0,1])$ such that $\mathcal{P}u = u$. Every such solution will also lie in $C^1([0,1])$ as can be directly shown by differentiating (2.3) and confirming the continuity.

To establish the existence and uniqueness to $\mathcal{P}u = u$, we show that the conditions of Theorem 2 hold. Consider the pair $(X,d) = (C[0,1],d)$ which forms a complete metric space. For $u_1,u_2 \in C([0,1])$ and $t \in [0,1]$, consider

$$\begin{aligned}
|(\mathcal{P}u_1)(t) - (\mathcal{P}u_2)(t)| \leq &\quad \int_0^1 |\mathcal{K}(t,\mathrm{s})||f(t,u_1(s)) - f(t,u_2(s))|ds \\
\leq &\quad M\int_0^1 |\mathcal{K}(t,\mathrm{s})||u_1(s)-u_2(s)|ds \\
\leq &\quad Md(u_1,u_2)\int_0^1 |\mathcal{K}(s,\mathrm{s})|ds \\
\leq &\quad M\int_0^1 |\mathcal{K}(s,\mathrm{s})|ds \;\; d(u_1,u_2).
\end{aligned}$$

Taking the maximum of both sides of the above inequality over $[0,1]$ we thus have for all $u_1,u_2 \in C([0,1])$,

$$d(Fu_1,Fu_2) \leq M\int_0^1 |\mathcal{K}(s,\mathrm{s})|ds \;\; d(u_1,u_2),$$

and in light of (3.1) we see that F satisfies all of the conditions of Theorem 2. Thus, the operator T has a unique fixed point in $C([0,1])$. This solution is also in $C^1([0,1])$ and we have equivalently shown that the BVP (1.4) has a unique (nontrivial) solution.

Theorem 3.2. Let M be a nonnegative constant such that

$$|f(t,\mathrm{u}) - f(t,\mathrm{v})| \leq M|\mathrm{u}-\mathrm{v}| \quad \forall \quad (t,\mathrm{u}),(t,\mathrm{v}) \in [0,1]\times\mathbb{R}.$$

Then the boundary value problem (1.4) is Hyers-Ulam stable if for each $\varepsilon > 0$, the solution $\hat{\mathrm{u}}(t)$ of the boundary value problem (1.4) satisfies the following inequality

$$\left|\hat{\mathrm{u}}(t) - \int_0^1 \mathcal{K}(t,\mathrm{s})f(\mathrm{s},\hat{\mathrm{u}}(\mathrm{s}))ds\right| < \varepsilon.$$

Proof. Let $\hat{\mathrm{u}}(t)$ be the unique solution of the problem (1.4). Then, we have

$$\begin{aligned}
|\hat{\mathrm{u}}(t) - \mathrm{z}(t)| = &\quad |\hat{\mathrm{u}}(t) - \int_0^1 \mathcal{K}(t,\mathrm{s})f(\mathrm{s},\hat{\mathrm{u}}(\mathrm{s}))ds \\
&\quad -\int_0^1 \mathcal{K}(t,\mathrm{s})f(\mathrm{s},\mathrm{u}(\mathrm{s}))ds + \int_0^1 \mathcal{K}(t,\mathrm{s})\mathrm{f}(\mathrm{s},\hat{\mathrm{u}}(\mathrm{s}))ds| \\
\leq &\quad \varepsilon + \int_0^1 \mathcal{K}(t,\mathrm{s})|\mathrm{f}(\mathrm{s},\hat{\mathrm{u}}(\mathrm{s})) - \mathrm{f}(\mathrm{s},\mathrm{u}(\mathrm{s}))|ds \\
\leq &\quad \varepsilon + M\int_0^1 |\mathcal{K}(\mathrm{s},\mathrm{s})||\hat{\mathrm{u}}(\mathrm{s}) - \mathrm{u}(\mathrm{s})|ds \\
\leq &\quad \varepsilon + M\int_0^1 |\mathcal{K}(s,\mathrm{s})|ds \,\|\, \hat{\mathrm{u}} - \mathrm{u} \,\|,
\end{aligned}$$

which implies that $\|\, \hat{\mathrm{u}} - \mathrm{u} \,\| \leq \varepsilon$, where $= \left[1 - M\int_0^1 |\mathcal{K}(s,\mathrm{s})|ds\right]^{-1}$. This completes the proof.

$$CD_{0^+}^{1.5u(t)=} \;\; f(t,u(t)),^{cu(1)= \;-du\prime(1),} \tag{3.2}$$

where

$$f(t,u(t)) = \frac{1}{(50e^{4t}+1)(1+|u(t)|)}.$$

It can be seen that all conditions of Theorem 3 and Theorem 3 are satisfied. Hence, the BVP (3.2) has a nontrivial unique solution and is Hyers-Ulam stable.

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