

Cartesian Product of IVDIFK – ideals in BCK/BCI-algebras

*R.Angelin Suba, #K.R.Sobha

*Research Scholar(part time internal),Reg.no- 18223182092001,
Sree Ayyappa College for Women, Chunkankadai. Nagercoil-629001,Tamilnadu, India
[Affiliated to Manonmaniam Sundaranar University, Abishekapatti,
Tirunelveli-627012, Tamilnadu,India]
#Assistant Professor, Sree Ayyappa College for Women,
Chunkankadai,Nagercoil.
Tamilnadu, India

ABSTRACT

The purpose of this paper is to define the notion of a Cartesian product of IVDIFK-ideals in BCK/BCI-algebras. Also we define I-V level K-ideals and necessary and sufficient condition for I-V level K-ideals in BCI/BCK-algebra are stated.

KEYWORDS

BCI-algebra, K- ideal, fuzzy K-ideal, intuitionistic fuzzy K-ideal, IVDIFK-ideal

I. INTRODUCTION

The notion of BCK-algebras was proposed by Imai and Iseki in 1966. In the same year, Iseki introduced the notion of a BCI-algebra which is a generalization of a BCK-algebra. Fuzzy sets were initiated by Zadeh. The idea of “intuitionistic fuzzy set” was first published by Atanassov as a generalization of the notion of fuzzy sets. After that many researchers considered the fuzzification of ideals and subalgebras in BCK(BCI)-algebras. In this paper, using the notion of interval valued fuzzy set, we introduce the concept of Cartesian product of IVDIFK-ideal in BCK/BCI-algebras and we prove that Cartesian product of two IVDIFK-ideals is again a IVDIFK-ideals.

II. PRELIMINARIES

In this section, some elementary aspects that are necessary for this paper are included.

An interval-valued intuitionistic fuzzy set (briefly, I-V IFS) A defined on X is given by

$$A = \{ (x, [\sigma_A^L(x), \sigma_A^U(x)], [\rho_A^L(x), \rho_A^U(x)]) \mid \forall x \in X \} \text{ (briefly, denoted by } A = [(\sigma_A^L, \sigma_A^U), (\rho_A^L, \rho_A^U)] \text{),}$$

Where σ_A^L, σ_A^U are two membership functions and ρ_A^L, ρ_A^U are two nonmembership functions in X such that $\sigma_A^L \leq \sigma_A^U$ and $\rho_A^L \geq \rho_A^U, \forall x \in X$. Let $\bar{\sigma}_A(x) = [\sigma_A^L, \sigma_A^U]$ and $\bar{\rho}_A(x) = [\rho_A^L, \rho_A^U], \forall x \in X$

and let $D[0,1]$ denote the family of all closed subintervals of $[0,1]$. If $\sigma_A^L(x) = \sigma_A^U(x) = c$,

$0 \leq c \leq 1$ and if $\rho_A^L(x) = \rho_A^U(x) = k, 0 \leq k \leq 1$, then we have $\bar{\sigma}_A(x) = [c, c]$ and $\bar{\rho}_A(x) = [k, k]$,

which we also assume, for the sake of convenience, to belong to $D[0,1]$. Thus $\bar{\sigma}_A(x)$ and

$\bar{\rho}_A(x) \in [0,1] \forall x \in X$, and therefore I-V IFS A is given by $A = \{ (x, \bar{\sigma}_A(x), \bar{\rho}_A(x)) \mid \forall x \in X \}$, where $\bar{\sigma}_A(x): X \rightarrow [0, 1], \bar{\rho}_A(x): X \rightarrow [0,1]$. Now let us define refined minimum, refined maximum (briefly $r \min, r \max$) of two

elements in $D[0,1]$. we also define the symbols " $\leq$ ", " $\geq$ " and " $=$ " in the case of two elements in $D[0,1]$. Consider two elements $D_1 = [a_1, b_1]$ and $D_2 = [a_2, b_2] \in D[0,1]$.

Then $r \min(D_1, D_2) = [\min\{a_1, a_2\}, \min\{b_1, b_2\}]$

$r \max(D_1, D_2) = [\max\{a_1, a_2\}, \max\{b_1, b_2\}]$

$D_1 \geq D_2 \Leftrightarrow a_1 \geq a_2, b_1 \geq b_2; D_1 \leq D_2 \Leftrightarrow a_1 \leq a_2, b_1 \leq b_2$

Definition 2.1 : An interval valued intuitionistic fuzzy set A in BCI algebra X is called an interval – valued doubt intuitionistic fuzzy K - ideal of X if it satisfies

- i) $\overline{\sigma}_A(0) \leq \overline{\sigma}_A(x)$, $\overline{\rho}_A(0) \geq \overline{\rho}_A(x)$
- ii) $\overline{\sigma}_A(x * (y * z)) \leq r \max\{\overline{\sigma}_A(x * (t * (y * z))), \overline{\sigma}_A(t)\}$
- iii) $\overline{\rho}_A(x * (y * z)) \geq r \min\{\overline{\rho}_A(x * (t * (y * z))), \overline{\rho}_A(t)\} \forall x, y, z, t \in X$

III. MAIN RESULTS

Definition 3.1 : An intuitionistic fuzzy relation A on any set A is a intuitionistic fuzzy subset A with a membership function $\Phi_A : X \times X \rightarrow [0,1]$ and non membership function $\Psi_A : X \times X \rightarrow [0,1]$

Lemma 3.2 : Let $\overline{\sigma}_A$ and $\overline{\sigma}_B$ be two membership functions and $\overline{\rho}_A$ and $\overline{\rho}_B$ be two non membership functions of each $x \in X$ to the doubt i-v subsets A and B , respectively. Then $\sigma_A \times \sigma_B$ is membership function and $\rho_A \times \rho_B$ is non membership function of each element $(x, y) \in X \times X$ to the set $A \times B$ and defined by

$$(\overline{\sigma}_A \times \overline{\sigma}_B)(x, y) = r \max\{\overline{\sigma}_A(x), \overline{\sigma}_B(y)\}, (\overline{\rho}_A \times \overline{\rho}_B)(x, y) = r \min\{\overline{\rho}_A(x), \overline{\rho}_B(y)\}$$

Theorem 3.3: Let $A = [(\sigma_A^L, \sigma_A^U), (\rho_A^L, \rho_A^U)]$ and $B = [(\sigma_B^L, \sigma_B^U), (\rho_B^L, \rho_B^U)]$ be two I-V doubt intuitionistic fuzzy K -ideals in a set X , then $A \times B$ is an I-V doubt intuitionistic fuzzy K -ideal of $X \times X$.

Proof : Let $(x, y) \in X \times X$ then by definition,

$$\begin{aligned} (\overline{\sigma}_A \times \overline{\sigma}_B)(0,0) &= r \max\{\overline{\sigma}_A(0), \overline{\sigma}_B(0)\} = r \max\{[\sigma_A^L(0), \sigma_A^U(0)], [\sigma_B^L(0), \sigma_B^U(0)]\} \\ &= [\max\{\sigma_A^L(0), \sigma_B^L(0)\}, \max\{\sigma_A^U(0), \sigma_B^U(0)\}] \leq [\max\{\sigma_A^L(x), \sigma_B^L(y)\}, \max\{\sigma_A^U(x), \sigma_B^U(y)\}] \\ &= r \max\{[\sigma_A^L(x), \sigma_A^U(x)], [\sigma_B^L(y), \sigma_B^U(y)]\} = r \max\{\overline{\sigma}_A(x), \overline{\sigma}_B(y)\} = (\overline{\sigma}_A \times \overline{\sigma}_B)(x, y) \\ \text{and } (\overline{\rho}_A \times \overline{\rho}_B)(0,0) &= r \min\{\overline{\rho}_A(0), \overline{\rho}_B(0)\} = r \min\{[\rho_A^L(0), \rho_A^U(0)], [\rho_B^L(0), \rho_B^U(0)]\} \\ &= [\min\{\rho_A^L(0), \rho_B^L(0)\}, \min\{\rho_A^U(0), \rho_B^U(0)\}] \geq [\min\{\rho_A^L(x), \rho_B^L(y)\}, \min\{\rho_A^U(x), \rho_B^U(y)\}] \\ &= r \min\{[\rho_A^L(x), \rho_A^U(x)], [\rho_B^L(y), \rho_B^U(y)]\} = r \min\{\overline{\rho}_A(x), \overline{\rho}_B(y)\} = (\overline{\rho}_A \times \overline{\rho}_B)(x, y) \end{aligned}$$

Now for all $x, y, z, t \in X$, we have

$$(\overline{\sigma}_A \times \overline{\sigma}_B)[(x, x') * ((y, y') * (z, z'))] = (\overline{\sigma}_A \times \overline{\sigma}_B)[(x, x') * (y * z, y' * z')] =$$

$$\begin{aligned}
&= (\bar{\sigma}_A \times \bar{\sigma}_B)[x * (y * z), x' * (y' * z')] = r \max\{\bar{\sigma}_A(x * (y * z)), \bar{\sigma}_B(x' * (y' * z'))\} \\
&\leq r \max\{r \max[\bar{\sigma}_A(x * (t * (y * z))), \bar{\sigma}_A(t)], r \max[\bar{\sigma}_B(x' * (t' * (y' * z'))), \bar{\sigma}_B(t')]\} \\
&= r \max\left\{\begin{aligned} &[\max\{\sigma_A^L(x * (t * (y * z))), \sigma_A^L(t)\}, \max\{\sigma_A^U(x * (t * (y * z))), \sigma_A^U(t)\}] \\ &[\max\{\sigma_B^L(x' * (t' * (y' * z'))), \sigma_B^L(t')\}, \max\{\sigma_B^U(x' * (t' * (y' * z'))), \sigma_B^U(t')\}] \end{aligned}\right\} \\
&= \begin{bmatrix} \max\{\max\{\sigma_A^L(x * (t * (y * z))), \sigma_B^L(x' * (t' * (y' * z')))\}, \max\{\sigma_A^L(t), \sigma_B^L(t')\}\}, \\ \max\{\max\{\sigma_A^U(x * (t * (y * z))), \sigma_B^U(x' * (t' * (y' * z')))\}, \max\{\sigma_A^U(t), \sigma_B^U(t')\}\} \end{bmatrix} \\
&= r \max\{(\bar{\sigma}_A \times \bar{\sigma}_B)[(x * (t * (y * z))), (x' * (t' * (y' * z')))], (\bar{\sigma}_A \times \bar{\sigma}_B)(t, t')\}
\end{aligned}$$

$$\text{Also, } (\bar{\rho}_A \times \bar{\rho}_B)[(x, x') * ((y, y') * (z, z'))] = (\bar{\rho}_A \times \bar{\rho}_B)[(x, x') * (y * z, y' * z')]$$

$$\begin{aligned}
&= (\bar{\rho}_A \times \bar{\rho}_B)[x * (y * z), x' * (y' * z')] = r \min\{\bar{\rho}_A(x * (y * z)), \bar{\rho}_B(x' * (y' * z'))\} \\
&\geq r \min\{r \min[\bar{\rho}_A(x * (t * (y * z))), \bar{\rho}_A(t)], r \min[\bar{\rho}_B(x' * (t' * (y' * z'))), \bar{\rho}_B(t')]\} \\
&= r \min\left\{\begin{aligned} &[\min\{\rho_A^L(x * (t * (y * z))), \rho_A^L(t)\}, \min\{\rho_A^U(x * (t * (y * z))), \rho_A^U(t)\}] \\ &[\min\{\rho_B^L(x' * (t' * (y' * z'))), \rho_B^L(t')\}, \min\{\rho_B^U(x' * (t' * (y' * z'))), \rho_B^U(t')\}] \end{aligned}\right\} \\
&= \begin{bmatrix} \min\{\min\{\rho_A^L(x * (t * (y * z))), \rho_B^L(x' * (t' * (y' * z')))\}, \min\{\rho_A^L(t), \rho_B^L(t')\}\}, \\ \min\{\min\{\rho_A^U(x * (t * (y * z))), \rho_B^U(x' * (t' * (y' * z')))\}, \min\{\rho_A^U(t), \rho_B^U(t')\}\} \end{bmatrix} \\
&= r \min\{(\bar{\rho}_A \times \bar{\rho}_B)[(x * (t * (y * z))), (x' * (t' * (y' * z')))], (\bar{\rho}_A \times \bar{\rho}_B)(t, t')\}
\end{aligned}$$

Hence $A \times B$ is an I-V doubt intuitionistic fuzzy K-ideal of $X \times X$.

Theorem 3.4 : Let A be an I-V doubt intuitionistic fuzzy set in a BCI-algebra X . Then A is an I-V doubt intuitionistic fuzzy K-ideal of X if and only if the non-empty sets

$$\Delta(A; [d_1, d_2]) =$$

$\{x \in X / \bar{\sigma}_A(x) \leq [d_1, d_2]\}$ and $\nabla(A; [i_1, i_2]) = \{x \in X / \bar{\rho}_A(x) \geq [i_1, i_2]\}$ are K-ideals of X for every $[d_1, d_2]$ and $[i_1, i_2] \in D[0, 1]$. We then call $\Delta(A; [d_1, d_2])$ and $\nabla(A; [i_1, i_2])$ are I-V level K-ideals of X .

Proof : Assume that A is an I-V doubt intuitionistic fuzzy K-ideal of X .

Since for all $x \in \Delta(A; [d_1, d_2])$ and $\nabla(A; [i_1, i_2])$,

We have $\bar{\sigma}_A(0) \leq \bar{\sigma}_A(x) \leq [d_1, d_2]$ and $\bar{\rho}_A(0) \geq \bar{\rho}_A(x) \geq [i_1, i_2]$

Therefore $0 \in \Delta(A; [d_1, d_2])$ and $\nabla(A; [i_1, i_2])$

Now let $x * (t * (y * z)) \& t \in \Delta(A; [d_1, d_2])$ and $\nabla(A; [i_1, i_2])$

Then, $\bar{\sigma}_A(x * (y * z)) \leq r \max\{\bar{\sigma}_A(x * (t * (y * z))), \bar{\sigma}_A(t)\} \leq r \max\{[d_1, d_2], [d_1, d_2]\} = [d_1, d_2]$

$\bar{\rho}_A(x * (y * z)) \geq r \min\{\bar{\rho}_A(x * (t * (y * z))), \bar{\rho}_A(t)\} \geq r \min\{[i_1, i_2], [i_1, i_2]\} = [i_1, i_2]$

And so $x * (y * z) \in \Delta(A; [d_1, d_2])$ and $\nabla(A; [i_1, i_2])$

Hence $\Delta(A; [d_1, d_2])$ and $\nabla(A; [i_1, i_2])$ are K-ideals of X.

Conversely assume that $x \in \Delta(A; [d_1, d_2])$ ($\neq \varphi$), $x \in \nabla(A; [i_1, i_2])$ ($\neq \varphi$) are K-ideals of X for every $[d_1, d_2] \in D[0,1]$, also $[i_1, i_2] \in D[0,1]$.

Suppose that there exist $x_0, y_0, z_0, t_0 \in X$ such that

$$\bar{\sigma}_A(x_0 * (y_0 * z_0)) > r \max \{ \bar{\sigma}_A(x_0 * (t_0 * (y_0 * z_0))), \bar{\sigma}_A(t_0) \} \text{ and}$$

$$\bar{\rho}_A(x_0 * (y_0 * z_0)) < r \min \{ \bar{\rho}_A(x_0 * (t_0 * (y_0 * z_0))), \bar{\rho}_A(t_0) \}$$

Let $\bar{\sigma}_A(x_0 * (t_0 * (y_0 * z_0))) = [\delta_1, \delta_2]$, $\bar{\sigma}_A(t_0) = [\delta_3, \delta_4]$ and $\bar{\sigma}_A(x_0 * (y_0 * z_0)) = [\lambda_1, \lambda_2]$ then

$$[\lambda_1, \lambda_2] > r \max \{ [\delta_1, \delta_2], [\delta_3, \delta_4] \} = [\max \{ \delta_1, \delta_3 \}, \max \{ \delta_2, \delta_4 \}]$$

Hence $\lambda_1 > \max \{ \delta_1, \delta_3 \}$ and $\lambda_2 > \max \{ \delta_2, \delta_4 \}$

$$\text{Taking } [\alpha_1, \alpha_2] = \frac{1}{2} [\bar{\sigma}_A(x_0 * (y_0 * z_0)) + r \max \{ \bar{\sigma}_A(x_0 * (t_0 * (y_0 * z_0))), \bar{\sigma}_A(t_0) \}]$$

$$\text{We obtain } [\alpha_1, \alpha_2] = \frac{1}{2} ([\lambda_1, \lambda_2] + [\max \{ \delta_1, \delta_3 \}, \max \{ \delta_2, \delta_4 \}])$$

$$= [\frac{1}{2}(\lambda_1 + \max \{ \delta_1, \delta_3 \}), \frac{1}{2}(\lambda_2 + \max \{ \delta_2, \delta_4 \})]$$

$$\text{It follows that, } \max \{ \delta_1, \delta_3 \} < \alpha_1 = \frac{1}{2}(\lambda_1 + \max \{ \delta_1, \delta_3 \}) < \lambda_1$$

$$\max \{ \delta_2, \delta_4 \} < \alpha_2 = \frac{1}{2}(\lambda_2 + \max \{ \delta_2, \delta_4 \}) < \lambda_2$$

$$\text{so that } [\max \{ \delta_1, \delta_3 \}, \max \{ \delta_2, \delta_4 \}] < [\alpha_1, \alpha_2] < [\lambda_1, \lambda_2] = \bar{\sigma}_A(x_0 * (y_0 * z_0))$$

Therefore $x_0 * (y_0 * z_0) \notin \Delta(A; [\alpha_1, \alpha_2])$

$$\text{On the other hand, } \bar{\sigma}_A(x_0 * (t_0 * (y_0 * z_0))) = [\delta_1, \delta_2] \leq [\max \{ \delta_1, \delta_3 \}, \max \{ \delta_2, \delta_4 \}] < [\alpha_1, \alpha_2]$$

$$\text{And } \bar{\sigma}_A(t_0) = [\delta_3, \delta_4] \leq [\max \{ \delta_1, \delta_3 \}, \max \{ \delta_2, \delta_4 \}] < [\alpha_1, \alpha_2]$$

$$\text{And so } x_0 * (t_0 * (y_0 * z_0)), t_0 \in \Delta(A; [\alpha_1, \alpha_2])$$

$$\text{And let } \bar{\rho}_A(x_0 * (t_0 * (y_0 * z_0))) = [\mu_1, \mu_2], \bar{\rho}_A(t_0) = [\mu_3, \mu_4] \text{ and } \bar{\rho}_A(x_0 * (y_0 * z_0)) = [\varepsilon_1, \varepsilon_2] \text{ then } [\varepsilon_1, \varepsilon_2] <$$

$$r \min \{ [\mu_1, \mu_2], [\mu_3, \mu_4] \} = [\min \{ \mu_1, \mu_3 \}, \min \{ \mu_2, \mu_4 \}]$$

$$\text{Hence } \varepsilon_1 < \min \{ \mu_1, \mu_3 \} \text{ and } \varepsilon_2 < \min \{ \mu_2, \mu_4 \}$$

$$\text{Taking } [\beta_1, \beta_2] = \frac{1}{2} [\bar{\rho}_A(x_0 * (y_0 * z_0)) + r \min \{ \bar{\rho}_A(x_0 * (t_0 * (y_0 * z_0))), \bar{\rho}_A(t_0) \}]$$

$$\text{We obtain } [\beta_1, \beta_2] = \frac{1}{2} ([\varepsilon_1, \varepsilon_2] + [\min \{ \mu_1, \mu_3 \}, \min \{ \mu_2, \mu_4 \}])$$

$$= [\frac{1}{2}(\varepsilon_1 + \min \{ \mu_1, \mu_3 \}), \frac{1}{2}(\varepsilon_2 + \min \{ \mu_2, \mu_4 \})]$$

$$\text{It follows that, } \min \{ \mu_1, \mu_3 \} > \beta_1 = \frac{1}{2}(\varepsilon_1 + \min \{ \mu_1, \mu_3 \}) > \varepsilon_1$$

$$\min \{ \mu_2, \mu_4 \} > \beta_2 = \frac{1}{2}(\varepsilon_2 + \min \{ \mu_2, \mu_4 \}) > \varepsilon_2$$

$$\text{so that } [\min \{ \mu_1, \mu_3 \}, \min \{ \mu_2, \mu_4 \}] > [\beta_1, \beta_2] > [\varepsilon_1, \varepsilon_2] = \bar{\rho}_A(x_0 * (y_0 * z_0))$$

Therefore $x_0 * (y_0 * z_0) \notin \nabla(A; [\beta_1, \beta_2])$

$$\text{On the other hand, } \bar{\rho}_A(x_0 * (t_0 * (y_0 * z_0))) = [\mu_1, \mu_2] \geq [\min \{ \mu_1, \mu_3 \}, \min \{ \mu_2, \mu_4 \}] > [\beta_1, \beta_2]$$

$$\text{And } \bar{\rho}_A(t_0) = [\mu_3, \mu_4] \geq [\min \{ \mu_1, \mu_3 \}, \min \{ \mu_2, \mu_4 \}] > [\beta_1, \beta_2]$$

$$\text{And so } x_0 * (t_0 * (y_0 * z_0)), t_0 \in \nabla(A; [\beta_1, \beta_2])$$

But this contradicts the fact that $\Delta(A; [d_1, d_2])$ and $\nabla(A; [i_1, i_2])$ are K-ideals of X.

Hence $\bar{\sigma}_A(x * (y * z)) \leq r \max \{ \bar{\sigma}_A(x * (t * (y * z))), \bar{\sigma}_A(t) \}$

$\bar{\rho}_A(x * (y * z)) \geq r \min \{ \bar{\rho}_A(x * (t * (y * z))), \bar{\rho}_A(t) \}$ for all $x, y, z, t \in X$.

REFERENCES

- 1 . K.T.Atanassov, Intuitionistic fuzzy sets, Fuzzy sets and systems, 1986; 20(10) : 87-96
- 2 . N.Palaniappan, P.S. Veerappan and R.Devi, Intuitionistic Fuzzy H-ideals of BCI-algebras with interval valued membership and Non membership Functions, Int . J. Contemp. Math. Sciences, Vol. 7, 2012, no. 43, 2125-2135
3. T. Bej and M.Pal, Doubt intuitionistic fuzzy ideals in BCK/BCI – algebras, Communicated.
4. K.Iseki, On BCI-algebras, Math. Sem. Notes Kobe Univ. 8 (1980) 125-130
5. K.Iseki and S. Tanaka, An Introduction to the theory of BCK – algebras, Math. Japon. 23(1) (1978/79) 1- 26
6. K.Iseki , An algebra related with a propositional calculus , Proc, Japan Acad.42 (1966), 26-29
7. T. Senapati, M. Bhowmik and M. Pal, Interval-valued intuitionistic fuzzy BG-subalgebras, The Journal of Fuzzy Mathematics, 20 (3) (2012), 707-720.
8. T. Senapati, M. Bhowmik and M. Pal, Interval-valued intuitionistic fuzzy closed ideals of BG-algebra and their products, Int. J. Fuzzy Logic Systems, 2 (2) (2012), 27-44.
9. L.A. Zadeh , Fuzzy sets, Inform. Control 8 (1965) 338-353